Crack propagation in viscoelastic solids

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We study crack propagation in a viscoelastic solid. Using simple arguments, we derive equations for the velocity dependence of the crack-tip radius, $a(v)$, and for the energy per unit area to propagate the crack, $G(v)$. For a viscoelastic modulus $E(\omega)$ which increases as ω^{1-s} $(0<s<1)$ in the transition region between the rubbery region and the glassy region, we find that $a(v)\sim G(v)\sim v^{\alpha}$ with $\alpha=(1-s)/(2-s)$. The theory is in good agreement with experiment.

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I. INTRODUCTION

The propagation of cracks in rubber is fundamental for many important applications—e.g., rubber wear [1]—and for pressure-sensitive adhesives [2]. The strength of the adhesion and cohesion of elastomers can be characterized by the amount of energy G required to advance a fracture plane by one unit area. Experiments have shown that G depends on the crack-tip velocity v and on the temperature T, and that [3–5]

$$G(v,T) = G_0[1 + f(v,T)], \tag{1}$$

where $f\to 0$ as $v\to 0$. Thus, G_0 is a threshold value below which no fracture occurs. The measured value of G at extremely low crack velocities and high temperatures, when viscous effects in the rubber are minimized, is of order ~ 1 eV/Å^2 and can be identified as G_0. At high crack velocities, G may be up to 10^4 times higher. For simple hydrocarbon elastomers, the effect of temperature can be accounted for completely by applying a simple multiplying factor a_T to the crack velocity v—i.e., $f(v,T)=f(a_T v)$. Moreover, values of a_T determined experimentally are equal to the Williams-Landel-Ferry [6] function determined from the temperature dependence of the bulk viscoelastic modulus. This proves that the large effects of crack velocity and temperature on crack propagation in rubber materials are due to viscoelastic processes in the bulk.

The energy dissipation at a crack in a viscoelastic solid has two contributions. The first is associated with the innermost region at the crack tip (the so-called crack tip process zone). It involves highly nonlinear processes [e.g., cavity formation, stringing, chain pull-out (for polymers), and bond breaking] and is described phenomenologically via the term $G_0=2\gamma_0$. (Note that for rubberlike materials γ_0 is much larger than the surface energy γ.) This contribution to $G(v)$ cannot be calculated reliably and is taken as an input (determined experimentally) in the theory. The second contribution comes from the viscoelastic dissipation in the polymer in the linear viscoelastic region in front of the tip. This contribution is calculated here, and we argue below that the exact form of the crack-tip process zone is not important for the calculation of the viscoelastic contribution to G.

In Sec. II we review briefly the standard theory of cracks in elastic solids. In Sec. III we calculate that the viscoelastic energy dissipation in the vicinity of a crack tip which propagates with the velocity v, and we derive a general expression for $G(v)=2\gamma_{\rm eff}(v)$. This quantity depends on a cutoff radius a that we determine in Sec. IV. Section V presents analytical and numerical results for $\gamma_{\rm eff}$ as a function of the crack-tip velocity v. The theoretical results are compared to experimental data in Sec. VI. In Sec. VII we comment on the more complex case of crack closing. A lot of work has been published on crack propagation in viscoelastic solids, and we compare our results to earlier studies in Sec. VIII. The energy-loss approach we use and the way we introduce the crack-tip radius a have several important advantages over the Barenblatt crack-tip model used in most earlier studies: (a) the analysis is much simpler, (b) the analysis can be extended to include the nonuniform temperature distribution at the crack tip, and (c) the analysis shows that the viscoelastic contribution to the crack-tip propagation energy G does not depend on the detailed processes which occur at the crack tip. This is very important, since these (highly complex) processes cannot be described accurately at present. In addition, (d) by using a simple sum rule for the viscoelastic modulus, we are able to simplify the derivation by avoiding the need to include the (complicated) angular dependence of the crack-tip stress field. Finally, (e) our treatment gives a simple closed formula for $G(v)$.

II. CRACK IN AN ELASTIC SOLID

In this paper we focus on cracks loaded in tension (mode I). We assume that the crack edge is parallel to the z axis; see Fig. 1. In elastic continuum theory, the stress in the vicinity

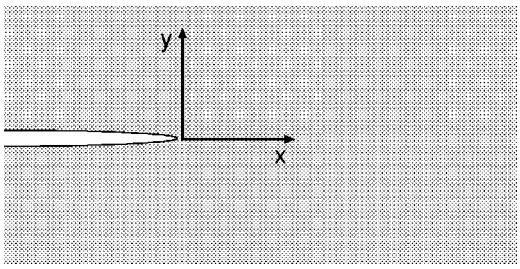

FIG. 1. Crack in an elastic solid. The crack edge is along the z axis.

of the crack tip has the universal form [7]

$$\sigma_{ij}(r,\phi) = K(2\pi r)^{-1/2} f_{ij}(\phi), \tag{2}$$

where the stress intensity factor K is proportional to the applied stress, but independent of the polar coordinates r and ϕ. The angular factor $f_{ij}(\phi)$ takes a universal form. The crack propagation energy G is given by [7]

$$G = \lambda K^2/E, \tag{3}$$

where E is the elastic modulus, and $\lambda = 1$ in plane stress and $\lambda = 1-\nu^2$ (where ν is the Poisson ratio) in plane strain. The crack propagation energy $G(v)$ does not depend on how the system is loaded, and in the following we focus on the simplest case of plane stress.

III. CRACK IN A VISCOELASTIC SOLID

Consider a crack loaded in tension (mode I) in a viscoelastic solid. We calculate the crack propagation energy $G(v)$, which is an intrinsic material quantity that does not depend on the geometry of the sample. Let us first calculate the energy dissipation per unit time and unit length of the crack line, P, for the general case of a crack propagating with velocity $\mathbf{v}$ in a linear viscoelastic solid. We have

$$P = \int d^2x \dot{\epsilon}_{ij}\sigma_{ij}, \tag{4}$$

where σ_{ij} is the stress tensor and $\dot{\epsilon}_{ij}$ the strain rate tensor. In continuum mechanics, for a homogeneous material, the general form of the stress in the vicinity of a crack tip is independent of the detailed relation between the stress and strain (i.e., also valid for a viscoelastic material) and takes the universal form (see Appendix A) [8]

$$\sigma(\mathbf{x},t) \approx K(2\pi|\mathbf{x}-\mathbf{v}t|)^{-1/2}. \tag{5}$$

Here we have ignored the tensorial aspect of the stress tensor which enters via a function $f_{ij}(\phi)$ that depends on the polar angle ϕ in the xy plane. We correct for this later. We write

$$\sigma(\mathbf{x},t) = \int d^2q \sigma(\mathbf{q},t) e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{6}$$

where

$$\begin{aligned}\sigma(\mathbf{q},t) &= \frac{1}{(2\pi)^2}\int d^2x e^{-i\mathbf{q}\cdot\mathbf{x}}|\mathbf{x}-\mathbf{v}t|^{-1/2}K' \\ &= \frac{K}{(2\pi)^{5/2}} e^{-iq_x v t}\int d^2x \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{|\mathbf{x}|^{1/2}} \\ &= \frac{K}{(2\pi)^{5/2}} e^{-iq_x v t} q^{-3/2}\int d^2w \frac{e^{-iw_x}}{|\mathbf{w}|^{1/2}}.\end{aligned}$$

We note that

$$\left|\int d^2w \frac{e^{-iw_x}}{|\mathbf{w}|^{1/2}}\right|^2 = 4\pi^2\left|\int_0^\infty dw w^{1/2} J_0(w)\right|^2 \equiv 4\pi^2\alpha,$$

where α is a number of order unity that also depends on the angular factor in the crack-tip stress distribution which we have neglected above. If we write

$$\sigma(\mathbf{q},t) = \sigma(q) e^{-iq_x v t}, \tag{7}$$

then

$$|\sigma(q)|^2 = \alpha K^2 q^{-3}/(2\pi)^3. \tag{8}$$

Substituting Eqs. (6) and (7) into Eq. (4) gives

$$\begin{aligned}P &= (2\pi)^2\int d^2q \frac{-iq_x v}{E(-q_x v)}|\sigma(q)|^2 \\ &= (2\pi)^2\int d\omega d^2q \delta(\omega + q_x v)\mathrm{Im}\frac{\omega}{E(\omega)}|\sigma(q)|^2,\end{aligned} \tag{9}$$

where $E(\omega)$ is the viscoelastic modulus of the solid. Substituting Eq. (8) into Eq. (9) gives

$$\begin{aligned}P &= \alpha K^2 \frac{1}{2\pi}\int d\omega d^2q \delta(\omega + q_x v)\mathrm{Im}\frac{\omega}{E(\omega)} q^{-3} \\ &= \alpha K^2 \frac{1}{\pi}\int_0^\infty d\omega\, \mathrm{Im}\frac{\omega}{E(\omega)}\int_0^\infty dq q^{-2}\int_0^{2\pi} d\phi \delta(\omega + qv\cos\phi).\end{aligned} \tag{10}$$

Since

$$\int_0^{2\pi} d\phi \delta(\omega + qv\cos\phi) = \frac{2\theta(qv-\omega)}{(q^2v^2-\omega^2)^{1/2}},$$

substituting into Eq. (10) gives

$$P = \alpha K^2 \frac{2}{\pi}\int_0^\infty d\omega\, \mathrm{Im}\frac{\omega}{E(\omega)}\int_{\omega/v}^{2\pi/a} dq q^{-2}(q^2v^2-\omega^2)^{-1/2}, \tag{11}$$

where we have introduced a large wave vector cutoff $2\pi/a$, where a may be a molecular distance (the distance between cross-links) or larger (see below). The q integral in Eq. (11) is easy to perform, yielding ($\omega = 2\pi v x/a$)

$$P = \alpha K^2 v \frac{2}{\pi}\int_0^1 dx \frac{(1-x^2)^{1/2}}{x}\,\mathrm{Im}\frac{1}{E(x\omega_c)}, \tag{12}$$

where $\omega_c = 2\pi v/a$. Now, let us consider the energy conservation condition relevant to the crack propagation. In the present case, the elastic energy stored in the solid in front of the crack tip is dissipated at the crack tip. The flow of elastic energy into the crack is given by Gv, which must equal the fracture energy term $G_0 v$ plus the bulk viscoelastic dissipation term P given by Eq. (12). Energy conservation gives

$$Gv = G_0 v + P. \tag{13}$$

Substituting Eq. (12) into Eq. (13) gives

$$G = G_0 + \alpha K^2 \frac{2}{\pi}\int_0^1 dx \frac{(1-x^2)^{1/2}}{x}\,\mathrm{Im}\frac{1}{E(x\omega_c)}. \tag{14}$$

Using $G = K^2/E(0)$ in Eq. (14) we obtain

$$G = \frac{G_0}{1-(2\alpha/\pi)E(0)\displaystyle\int_0^1 dx \frac{(1-x^2)^{1/2}}{x}\,\mathrm{Im}\frac{1}{E(x\omega_c)}}. \tag{15}$$

We can determine α by considering $\gamma_{\rm eff}$ for very large crack velocities; one can show that (Appendix A) [9]

$$G = G_0 \frac{E(\infty)}{E(0)}. \tag{16}$$

We can use this result to show that $\alpha=1$. In Appendix B we prove the *sum rule*

$$\frac{1}{E(0)} - \frac{1}{E(\infty)} = \frac{2}{\pi}\int_0^\infty d\omega \frac{1}{\omega} \,\mathrm{Im}\, \frac{1}{E(\omega)}. \tag{17}$$

When $v \to \infty$, we have $\omega_c \to \infty$ and

$$E_0 \frac{2}{\pi}\int_0^1 dx \frac{(1-x^2)^{1/2}}{x} \,\mathrm{Im}\, \frac{1}{E(x\omega_c)} \to E_0 \frac{2}{\pi}\int_0^\infty d\omega \frac{1}{\omega} \,\mathrm{Im}\, \frac{1}{E(\omega)}$$
$$= 1 - \frac{E(0)}{E(\infty)} \equiv \kappa. \tag{18}$$

If we substitute this result in Eq. (14) and choose $\alpha=1$, Eq. (16) is satisfied.

IV. TIP BLUNTING

Equation (14) depends on the cutoff length a, and Eq. (14) is of limited practical importance unless we have a way of determining this length. We determine a as follows. Experiments have shown that the crack-tip radius in polymers increases with increasing speed of the crack tip (see below). We choose a equal to the radius of the crack tip, which we determine as follows. The stress at the crack tip must be equal to the stress necessary to break the atomic bonds at the tip in order for the tip to propagate. If σ_c denotes this stress, which is a characteristic property of the material in question, we obtain, from Eq. (5),

$$\sigma_c = K/(2\pi a)^{1/2}, \tag{19}$$

where a depends on the crack-tip velocity. Combining this result with

$$G = K^2/E(0)$$

yields

$$G = \frac{2\pi a \sigma_c^2}{E(0)}. \tag{20}$$

Combining Eqs. (14) and (20) leads to

$$a = \frac{a_0}{1 - E_0 \frac{2}{\pi}\int_0^1 dx \frac{(1-x^2)^{1/2}}{x} \,\mathrm{Im}\, \frac{1}{E(x\omega_c)}}, \tag{21}$$

where $\omega_c = 2\pi v/a$ and where $a_0 = E(0)G_0/(2\pi\sigma_c^2)$. Since ω_c depends on a, this is an implicit equation for $a=a(v)$. Thus the present theory gives both the (velocity-dependent) radius of the crack tip, $a(v)$, and the crack propagation energy $G(v)=G_0 a(v)/a_0$.

For high crack-tip velocities, because of the high deformation frequencies at the crack tip (of order v/a), the rubber close to the tip is in an effectively hard glassy state. This is included in our treatment. The appearance of the zero-frequency elastic modulus $E(0)$ in the definition of the crack-tip radius a_0 follows from the fact that, in limit of zero crack propagation velocity, the perturbing frequency v/a_0 vanishes and only $E(0)$, and not the elastic modulus at some higher frequency, can enter in a_0.

Let us consider a general case where

$$\frac{1}{E(\omega)} = \frac{1}{E_\infty} + \int_0^\infty d\tau \frac{H(\tau)}{1 - i\omega\tau}, \tag{22}$$

where the *spectral density* $H(\tau)$ is real and positive. As shown in Appendix B, any causal linear response function can be represented in the form (22). Note that

$$\frac{1}{E_0} - \frac{1}{E_\infty} = \int_0^\infty d\tau H(\tau) \tag{23}$$

and

$$\mathrm{Im}\, \frac{1}{E(\omega)} = \int_0^\infty d\tau \frac{H(\tau)\omega\tau}{1 + (\omega\tau)^2}. \tag{24}$$

Substitution into Eq. (21) leads to

$$\frac{a}{a_0} = \frac{1}{1 - \kappa \int_0^\infty d\tau \hat{H}(\tau)\{[1 + b^{-2}(\tau)]^{1/2} - b^{-1}(\tau)\}}, \tag{25}$$

where $b = \omega_c\tau = 2\pi v\tau/a = (2\pi v\tau/a_0)(a_0/a)$ and

$$\hat{H}(\tau) = \frac{H(\tau)}{\int_0^\infty d\tau' H(\tau')}. \tag{26}$$

Note that $a/a_0 = G/G_0 = \gamma_{\rm eff}/\gamma_0$. This result and Eq. (25) are the main results of this paper.

Let us now consider the limit of large crack-tip velocity. When $v \to \infty$, $a/a_0 \to \infty$, but slower than the velocity v. Thus, for a fixed τ, $b \to \infty$ as $v \to \infty$. If we introduce a cutoff time $\tau = \tau^*$ via the condition $b(\tau^*)=1$ or $\tau^* = a/(2\pi v)$, we find, for large v,

$$\int_0^\infty d\tau \hat{H}(\tau)[(1 + b^{-2})^{1/2} - b^{-1}] \approx 1 - \int_{\tau^*}^\infty d\tau \hat{H}(\tau)\frac{1}{b}. \tag{27}$$

Equation (25) then takes the form

$$\left(\frac{G}{G_0}\right)^2 = \left(\frac{a}{a_0}\right)^2 = \left(\frac{a_0}{2\pi v}\int_{\tau^*}^\infty d\tau \frac{\hat{H}(\tau)}{\tau}\right)^{-1}. \tag{28}$$

Here we have put $\kappa=1$ (typically $E_0/E_\infty \approx 0.001$ so that $\kappa \approx 0.999$); the deviation of κ from unity becomes important only for very high velocities (see below).

V. ANALYTICAL AND NUMERICAL RESULTS FOR REALISTIC RUBBER MODELS

We assume that $H(\tau) = A\tau^{-1/2}$, which follows in the Rouse model [12] when $1/\tau_1 > \omega > 1/\tau_0$ (τ_0 is the so-called en-

tanglement time). For rubber compounds used for pressure-sensitive adhesives one has typically $\tau_0 \approx 1$ s and $\tau_1 \approx 10^{-6}$ s, giving $\tau_0/\tau_1 \approx 10^6$. For $\tau < \tau_1$ and $\tau > \tau_0$ we take $H(\tau)=0$. Now, since

$$\int_0^\infty d\tau H(\tau) = \int_{\tau_1}^{\tau_0} d\tau A \tau^{-1/2} \approx 2A\tau_0^{1/2},$$

for $\tau > \tau_1$ we get

$$\hat{H} \approx \frac{1}{2\tau_0}\left(\frac{\tau_0}{\tau}\right)^{1/2}.$$

Substituting this result in Eq. (28) gives

$$\frac{a}{a_0} = \left(\frac{v}{v_0}\right)^{1/3},$$

where $v_0 = a_0/(2\pi\tau_0)$. The effective interfacial energy for large v is

$$\gamma_{\rm eff} = \gamma_0 \frac{a}{a_0} = \gamma_0 \left(\frac{v}{v_0}\right)^{1/3}.$$

Since $\gamma_{\rm eff} \to \gamma_0$ as $v \to 0$, we can write the interpolation formula

$$\gamma_{\rm eff} = \gamma_0 \left(1 + \frac{v}{v_0}\right)^{1/3}, \tag{29}$$

which has the correct limiting behavior for large and small crack velocities. Equation (29) is valid only if v is below some value v_1, which can be determined by the fact that the maximum possible $\gamma_{\rm eff}$ is $\gamma_0 E(\infty)/E(0)$. This gives

$$\gamma_0 \left(\frac{v_1}{v_0}\right)^{1/3} = \gamma_0 \frac{E(\infty)}{E(0)}$$

or

$$v_1 \approx v_0 \left(\frac{E(\infty)}{E(0)}\right)^3.$$

In a typical case, $E(\infty)/E(0)$ equals 10^3–10^4, so that $v_1 \sim (10^9$–$10^{12})v_0$ in good agreement with experiment [5]. The relaxation time τ_0 can be identified with the so-called entanglement time, which for polymers used for pressure-sensitive adhesives typically (at room temperature) is of order ~ 1 s. Thus, the velocity $v_0 = a_0/(2\pi\tau_0)$ is typically of the order 10^{-10} m/s, which again agrees with rubber tear or peel experiments.

In Fig. 2 we compare the predictions for the velocity dependence of the effective surface energy $\gamma_{\rm eff}$ using the approximate equation (29) (dashed curve) with the full theory, Eq. (25) (solid curve).

Let us consider a more general viscoelastic modulus where $H(\tau) = A\tau^{-s}$ and $0 < s < 1$. For $s = 1/2$ this reduces to the case studied above. For many rubber compounds a good description of the viscoelastic properties in the transition region, between the rubbery and glassy regions, is obtained if s is slightly larger than 0.5, typically in the range from 0.5 to 0.7. As an example, we show in Fig. 3 (on a log-log scale) the Fourier transform $E(t)$ of the viscoelastic modulus $E(\omega)$

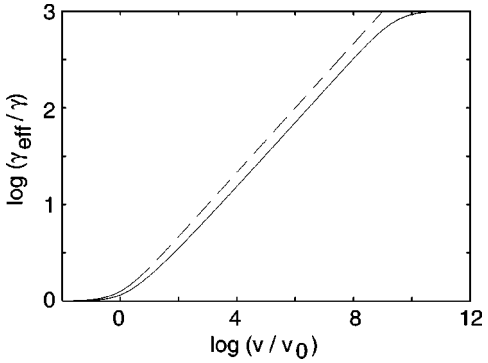

FIG. 2. The dependence of the effective surface energy $\gamma_{\rm eff}$ on the crack velocity v for a rubber characterized by the spectral density $H(\tau) \sim \tau^{-1/2}$ for $\tau_1 < \tau < \tau_0$ and zero otherwise. The solid line is the result of the full theory, Eq. (25), while the dashed line is the approximation (29). We have assumed $\tau_0/\tau_1 = 10^6$ and $E_\infty/E_0 = 10^3$. The reference velocity $v_0 = a_0/(2\pi\tau_0)$, where a_0 is the cutoff distance in the limit of arbitrary slowly moving crack. The logarithm is with 10 as the basis.

for a styrene-butadiene rubber compound filled with carbon black. The data are fitted well by a power law $E(t) \sim t^{-1.4}$ corresponding to $s = 0.6$ (since the exponent is given by $s - 2$). We obtain

$$\hat{H} = \frac{(1-s)}{\tau_0}\left(\frac{\tau_0}{\tau}\right)^s.$$

Substituting this in Eq. (28) gives, after some simplifications,

$$\frac{a}{a_0} = \left(\frac{v}{v_0}\right)^\alpha \left(\frac{s}{1-s}\right)^\beta, \tag{30}$$

where $\alpha = (1-s)/(2-s)$ and $\beta = 1/(2-s)$. Thus, for example, if $s = 0.6$, $a \sim \gamma_{\rm eff} \sim v^{2/7}$.

Finally, in Appendix C we study crack propagation in a viscoelastic material described by the so-called Kelvin model. This is not realistic for real rubber, but is often used in model calculations.

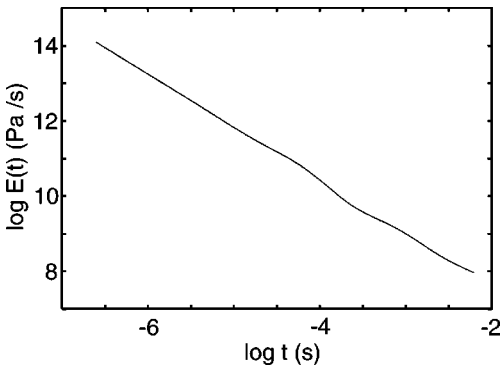

FIG. 3. The logarithm of the Fourier transform of the viscoelastic modulus as a function of the logarithm of time for a styrene-butadiene rubber compound.

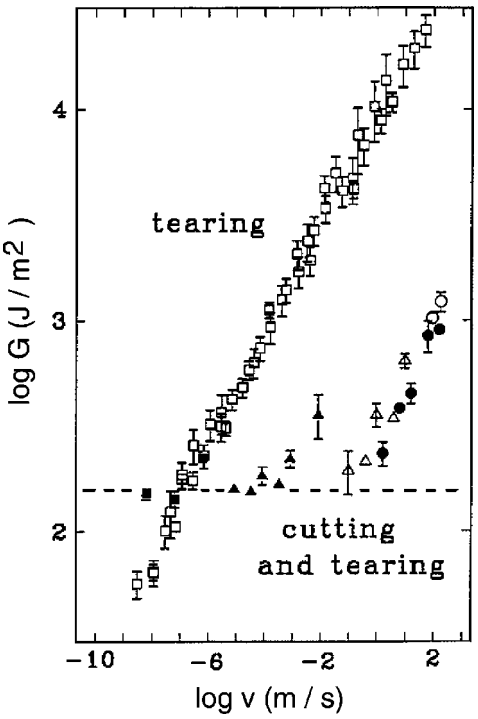

FIG. 4. Fracture energy G for styrene-butadiene rubber at various cutting or tearing speeds at $T=25$ °C. Adopted from Ref. [5].

VI. COMPARISON WITH EXPERIMENT

Figure 4 shows the fracture energy G for a styrene-butadiene copolymer rubber as a function of the crack tip velocity [5]. The results have been obtained using the Williams-Landel-Ferry [6] velocity-temperature shifting procedure, where the original data were obtained for crack velocities below 1 cm/s in order to reduce the influence of the crack-tip flash temperature. On the log-log scale the experimental data exhibit a straight line corresponding to a velocity dependence $G\sim v^{\alpha}$ with $\alpha\approx 0.27$, in good agreement with the theory presented above [for $s=0.6$ we have $\alpha=(1-s)/(2-s)\approx 0.28$]. Figure 4 also presents results obtained when rubber is torn apart while a sharp razor blade is pushed into the crack tip. Cutting resistance at low speeds is rather independent of speed, although at high speeds it increases markedly. In this case the crack-tip diameter is presumably given by the blade-tip diameter (about 1000 nm), which may explain the approximate independence of the fracture energy on crack velocity for small velocities.

VII. COMMENT ON CLOSING CRACKS IN VISCOELASTIC SOLIDS

The problem of crack growth studied above has many direct applications, including wear and adhesives. The opposite situation of crack closure in a viscoelastic media is important during the formation of contacts between a viscoelastic solid and another solid, as in standard adhesion tests where a rubber ball is brought into contact with a hard flat surface. However, in many practical situations, irreversible surface processes occur so that, even when crack propagation occurs very slowly, the energy (per unit area) $2\gamma_0$ for bond breaking during crack opening differs from the energy gain (per unit area) $2\gamma_0'$ due to bond formation during crack closing, with $\gamma_0\geqslant\gamma_0'$. However, when only the weak van der Waals interaction occurs between the crack planes, as is the case when a rubber ball is in contact with a glass surface, one may expect $\gamma_0\approx\gamma_0'$, and we will assume $\gamma_0=\gamma_0'$ in the following.

For a closing crack, the interfacial binding energy (or surface energy) $2\gamma_0$ is converted partly into elastic energy and partly dissipated in the rubber bulk, so that the energy conservation condition takes the form

$$\frac{dU_{\rm surf}}{dt}=\frac{dU_{\rm el}}{dt}+P,$$

and it is clear from energy conservation that $\gamma_{\rm eff}\leqslant\gamma_0$. Thus, while $\gamma_{\rm eff}$ may increase during crack growth by a factor of $\sim 10^4$ as the velocity increases, $\gamma_{\rm eff}\leqslant\gamma_0$ during crack closure. This large hysteresis in the adhesion energy, $\gamma_{\rm eff}$, has been observed experimentally [10] and may give an important contribution to the friction force during the sliding of a rubber block on a smooth asperity, as the process can be viewed as the contribution of a closing crack at the front edge of the contact region and a opening crack at the trailing edge [11].

VIII. DISCUSSION AND COMPARISON WITH OTHER STUDIES

Classical fracture mechanics is based on continuum mechanics and predicts a stress singularity at a crack tip, $\sigma\sim r^{-1/2}$, where r is the distance from the tip. However, any real material will yield when the stress becomes sufficiently high. In an ideal brittle material, such as mica, the relation $\sigma\sim r^{-1/2}$ may hold until $r\sim a$ is of the order of a lattice constant a. However, in most materials the $\sigma\sim r^{-1/2}$ relation will break down at much larger distances r. The spatial region in the vicinity of a crack tip where the relation $\sigma\sim r^{-1/2}$ is no longer valid is called the *crack-tip process zone*.

The crack propagation energy $G=2\gamma_{\rm eff}$ will generally depend on the exact nature of the processes occurring in the crack-tip process zone. Since these bond-breaking processes may be highly complex—e.g., involving cavity formation and stringing—the crack propagation energy cannot in general be calculated theoretically but must be deduced from experimental data. Only in the limiting case of ideal brittle solids (e.g., mica) will the crack propagation energy $G\approx 2\gamma$ be (nearly) equal to the energy per unit area required to break the atomic bonds at the (atomically sharp) crack tip. The surface energy γ is known for many solids and can sometimes can be calculated using electronic structure methods.

The standard model used to describe the crack tip process zone is due to Barenblatt [13], who assumed that the bond breaking at the crack tip occurs by stretching the bonds orthogonal to the crack surfaces until they break at some characteristic stress level σ_c. The process zone extends a distance a in front of the crack tip as indicated by the horizontal white line in Fig. 5(b). This model was first applied to crack propagation in viscoelastic solids by Schapery [14] and later by

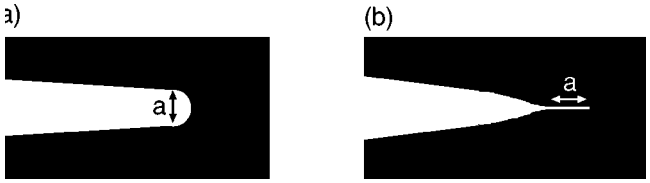

FIG. 5. The singular stress region at a crack tip in continuum mechanics can be removed either by (a) tip blunting (tip diameter a) or (b) by introducing a lateral region (linear size a) over which the bond breaking occurs. The latter is the so-called Barenblatt process zone.

Greenwood and Johnson [15], Barber *et al.* [16], and Hui and co-workers [17]. The present treatment introduces the cutoff in a more *ad hoc* manner, which may be roughly interpreted as a blunting of the crack tip as in Fig. 5(a). However, the exact way of introducing the cutoff is unimportant, and in reality the process zone is much more complex than assumed in the theory; see Fig. 6. In general, the cutoff should be introduced to simplify the analytical calculations as much as possible, and for crack propagation in viscoelastic solids we believe that our procedure results in the simplest formalism.

Barber, Donley, and Langer [16] studied crack propagation in viscoelastic solids using the Barenblatt model, which resulted in a very complex set of equations. The authors where nevertheless able to extract the high-velocity behavior of G, which agrees exactly with our limiting behavior given by Eq. (30). This limiting behavior was also obtained by Greenwood and Johnson [15], enforcing that the exact way the short-distance cutoff is introduced is unimportant. Only the factor G_0 in Eq. (1) depends on the crack-tip process zone. For polymers this quantity cannot be calculated accurately at present, and G_0 must be deduced directly from experimental data.

In reality, G_0 will also depend on the crack-tip velocity (and the temperature), although more weakly than the factor $f(v,T)$. This is because cavity formation, stringing, chain pull-out, and bond breaking all depend on the speed with which the surfaces are separated at the crack tip and on the temperature.

Our theoretical study shows that the crack propagation energy G indeed has the form given by Eq. (1), and we determine the function $f(v,T)$. The prefactor G_0 corresponds to the energy per unit area to pull-out and break the polymer chains at the crack tip. This energy is much larger than the surface energy of normal rubber, which is only a few meV/Å^2. In general, G_0 increases with increasing chain length, and in experiments probing the adhesive strength of partly cross-linked rubber sheets, G_0 appears to be directly proportional to the density of interfacial bonds. These results are both expected and supported by simple model calculations; see Ref. [18].

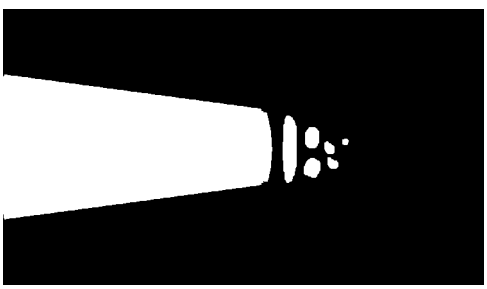

FIG. 6. The crack-tip process zone in most materials is very complex, involving cavity formation, stringing, chain pull-out (for polymers), and bond breaking.

The treatment presented above neglects the influence of the inhomogeneous temperature distribution, which occurs in the vicinity of the crack tip as a result of the inhomogeneous energy dissipation. This temperature effect should be extremely important for fast-moving cracks, but the measurements of Gent and others consider only relatively slowly moving cracks, $v<1$ cm/s. However, detailed theoretical calculations are necessary in order to determine under exactly which conditions temperature effects become important. In the context of tires sliding on road surfaces, flash temperature effects in the rubber may already be important for sliding velocities >0.1 cm/s [19].

IX. SUMMARY AND CONCLUSION

In this paper we have studied crack propagation in linear viscoelastic solids. We have focused mainly on crack opening, but we also considered the more complex (but less important) case of crack closing. We have calculated the dependence of the fracture energy (per unit area) $G=2\gamma_{\mathrm{eff}}$ on the crack velocity v. Our approach is based on energy conservation and is much simpler than the standard approach based on the Barenblatt model of the crack-tip process zone. Nevertheless, the two models give similar results in spite of the very different treatment of the crack-tip process zone. This suggests that the exact nature or shape of the crack-tip process zone is not important for the velocity dependence of $G(v)$, provided that the size of the process zone $a(v)$ increases with increasing tip velocity v in such a way that the stress at the crack tip does not exceed the critical value σ_c for bond breaking. This is an important result, since neither model treats the crack-tip process zone accurately (very complex processes will occur at the crack tip involving, e.g., cavity formation, stringing, chain pull-out, and bond breaking).

The treatment of crack propagation in viscoelastic solids presented in this paper can be extended to include the crack-tip flash temperature, which, because of the low heat conductivity of rubber materials, is likely to be of extreme importance already at relative low crack-tip velocities. We shall report on this study in another publication.

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APPENDIX A

In this appendix we explain the origin of two facts we have made use of in Sec. III. Neglecting inertia effects, the stress tensor satisfies

$$\sigma_{ij,j} = 0. \tag{A1}$$

In addition, the stress tensor must satisfy certain compatibility conditions, which, as long as the relation between the stress and strain is linear and the material homogeneous and isotropic, are independent of the constitutive relation between the stress and strain. For the plane stress or strain case which interests us here (where the stress tensor is independent of z), the compatibility equation becomes

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0. \tag{A2}$$

Note that Eqs. (A1) and (A2) constitute three independent equations for three unknown quantities: namely σ_{xx}, σ_{yy}, and σ_{xy}. It follows that the stress distribution in the vicinity of a crack tip has the universal form $\sim r^{-1/2}$ independent of the detailed form of the constitutive relation between stress and strain as long as the relation is linear and the material homogeneous and isotropic; i.e., it is also valid for a viscoelastic solid.

We now prove that during crack opening,

$$\lim_{v\to\infty} \gamma_{\text{eff}}(v) = \gamma_0 E_\infty/E_0. \tag{A3}$$

This relation is not valid during crack closing. The stress in the vicinity of the crack tip is of the form

$$\sigma = K(2\pi r)^{-1/2}, \tag{A4}$$

where the stress intensity factor K is proportional to the external applied stress. Assume that the crack tip propagates with a velocity v. The deformation rate of the viscoelastic solid a distance r from the crack tip is characterized by the frequency $\omega = v/r$. The smallest possible r is a lattice constant a, so that the highest possible frequency will be v/a. For very low velocities this frequency will be in the rubbery region of the viscoelastic spectra $E(\omega)$, and in this case the solid will behave purely elastically everywhere with the elastic modulus $E_0 = E(0)$. In this case there will be no dissipation in the bulk, and the crack propagation energy $G = 2\gamma_0$.

Next consider very high crack velocities v. For small enough r, the frequency ω will be so high that the rubber response will correspond to the glassy region where the elastic modulus is $E(\omega) \approx E_\infty$. On the other hand, when r is large enough the frequency $\omega = v/r$ will correspond to the rubbery region where $E(\omega) \approx E_0$. At intermediate distances $E(\omega)$ is complex and this "dissipative" region is indicated by the dark gray area in Fig. 7. The crack propagation energy G for an elastic medium is related to the stress intensity factor K via

$$G = K^2/E.$$

We first apply this formula to the inner region at the crack tip. In this case $G = 2\gamma_0$ and $E = E_\infty$, giving

$$2\gamma_0 = K^2/E_\infty. \tag{A5}$$

When we study the system at a lower magnification we do not observe the inner region and the dissipative region but only the outer region. In this case we must include in the crack propagation energy the energy dissipation in the rubber in the transition region (dark gray area in Fig. 7). Thus, G will now be larger than $2\gamma_0$, and we write for the outer region $G = 2\gamma_{\text{eff}}$. Since $E = E_0$ in the outer region, we get

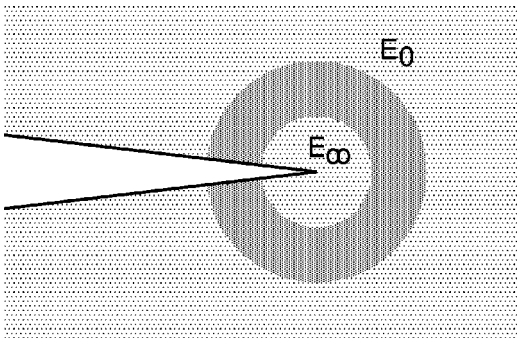

FIG. 7. When a crack propagates fast in a viscoelastic solid, one can distinguish between three separate spatial regions: (a) an inner region where the perturbing frequencies $\omega = v/r$ (where r is the distance from the crack tip) are so high that the rubber response corresponds to the hard glassy region characterized by the (high-frequency) elastic modulus E_∞, (b) an outer region where the perturbing frequencies are so small that the rubber responds with its zero-frequency modulus E_0, and (c) an intermediate region where the full complex viscoelastic modulus $E(\omega)$ enters and where the bulk viscoelastic energy dissipation occur (schematic).

$$2\gamma_{\text{eff}} = K^2/E_0. \tag{A6}$$

Combining Eqs. (A5) and (A6) gives (for $v \to \infty$)

$$\gamma_{\text{eff}} = \gamma_0 E_\infty/E_0.$$

APPENDIX B

The viscoelastic modulus $E(\omega)$ and the inverse $1/E(\omega)$ are causal linear response functions. Causality implies, for example, that the strain $\epsilon(t)$ in a solid at time t only depends on the stress $\sigma(t')$ it was exposed to at earlier times $t' \leqslant t$—i.e.,

$$\epsilon(t) = \int_{-\infty}^{t} dt' C(t-t')\sigma(t'). \tag{B1}$$

Defining the Fourier transform

$$\epsilon(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \epsilon(t) e^{i\omega t},$$

we get, from Eq. (B1),

$$\epsilon(\omega) = \sigma(\omega)/E(\omega),$$

where

$$\frac{1}{E(\omega)} = \int_0^\infty dt C(t) e^{i\omega t}.$$

Since $\text{Re}[i\omega t] < 0$ for $t > 0$ and $\text{Im}\,\omega > 0$, it follows that $1/E(\omega)$ is an analytical function of ω in the upper half of the complex frequency plane. Thus all poles and branch cuts of $1/E(\omega)$ will occur in the lower part of the complex ω plane and we may write

$$\frac{1}{E(\omega)} = \frac{1}{E_\infty} + \int_0^\infty d\tau \frac{H(\tau)}{1 - i\omega\tau}, \tag{B2}$$

where the *spectral density* $H(\tau)$ is real and positive. Using Eq. (B2) one can easily prove the *sum rule*

$$\frac{1}{E(0)} - \frac{1}{E(\infty)} = \frac{2}{\pi}\int_0^\infty d\omega \frac{1}{\omega} \,\mathrm{Im}\, \frac{1}{E(\omega)}. \tag{B3}$$

One can also derive Kramers-Kronig relations, relating the real part of $1/E(\omega)$ to the imaginary part of $1/E(\omega)$, but they are not needed in the present paper.

APPENDIX C

In this appendix we consider a very simple viscoelastic model where the "rubber" is characterized by a single relaxation time τ_0. Real rubber has instead a wide distribution of relaxation times, so the present model is not a good description of real rubber materials, but is nevertheless often used in model calculations. We assume

$$H(\tau) = \left(\frac{1}{E_0} - \frac{1}{E_\infty}\right)\delta(\tau - \tau_0),$$

so that

$$\hat{H}(\tau) = \delta(\tau - \tau_0).$$

Substituting this result in Eq. (25) gives

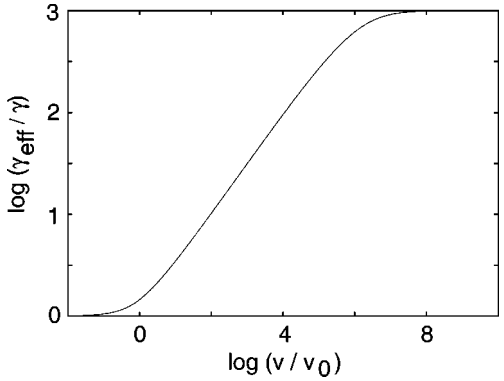

FIG. 8. The dependence of the effective surface energy $\gamma_{\rm eff}$ on the crack velocity v for a case where the rubber is characterized by a single relaxation time $\tau = \tau_0$. The reference velocity $v_0 = a_0/(2\pi\tau_0)$, where a_0 is the cutoff distance in the limit of an arbitrary slowly moving crack.

$$a = \frac{a_0}{1 - \kappa\{[1 + b^{-2}(\tau_0)]^{1/2} - b^{-1}(\tau_0)\}}, \tag{C1}$$

where $b(\tau_0) = (a_0/a)(2\pi v\tau_0/a_0) = (a_0/a)(v/v_0)$ where $v_0 = a_0/(2\pi\tau_0)$. In Fig. 8 we show $\gamma_{\rm eff}/\gamma_0 = a/a_0$ as a function of v/v_0 as obtained from Eq. (C1). Note that for intermediate crack velocities $10v_0 < v < 10^5 v_0$, $\gamma_{\rm eff} \sim a \sim v^{1/2}$.

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